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Some Extremal Signatures for Polynomials

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1. INTRODUCTION

Let V be an n-dimensional subspace of the space $C(D)$ of continuous real-valued functions on a compact Hausdorff space D. For the norm of $f \in C(D)$, we take

$$\|f\| = \sup\{|f(x)| : x \in D\}.$$

A *signature* σ is a function on D which has finite support and whose nonzero values are either $+1$ or -1. We say that a signature σ is *extremal* with respect to V if there exists a nonzero positive measure μ whose carrier is contained in the support of σ such that $\int u(x)\, \sigma(x)\, d\mu(x) = 0$ for all $u \in V$. By a convexity argument [1, 2], one can show that this definition is equivalent to the statement that there is no $u \in V$ such that $u(x)\, \sigma(x) > 0$ for all x in the support of σ.

The notion of an extremal signature plays a central role in the theory of Chebyshev approximation [1]. Indeed, suppose $f \in C(D)$ is given, and for each $u \in V$ define

$$E_f^+(u) = \{x \in D : f - u = \|f - u\|\},$$
$$E_f^-(u) = \{x \in D : f - u = -\|f - u\|\},$$
$$E_f(u) = E_f^+(u) \cup E_f^-(u).$$

The best approximations of f out of V are characterized in

THEOREM 1.1. [1]. *Let $d = \inf\{\|f - u\| : u \in V\}$. Then $u_* \in V$ satisfies $\|f - u_*\| = d$ if and only if there is an extremal signature σ with support in $E_f(u_*)$ such that $(f - u_*)\, \sigma \geq 0$.*

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As pointed out by Collatz [4], extremal signatures can be used to obtain lower bounds on the deviation of best Chebyshev approximations. The idea here is based on

THEOREM 1.2 [4]. *Let σ be an extremal signature with support E. Suppose $v \in V$ is such that $(f - v)\,\sigma \geqslant 0$. Then*

$$\min\{|f(x) - v(x)| : x \in E\} \leqslant d \leqslant \|f - v\|$$

where $d = \inf\{\|f - u\| : u \in V\}$.

If D is a subset of the real line and V satisfies the Haar condition, then a complete characterization of the extremal signatures is known [2]. Moreover, based on this characterization, effective numerical algorithms have been developed for computing best approximations [8]. However, for subspaces without the Haar condition, much less is known about the extremal signatures. For polynomials of several variables, a number of results concerning the extremal signatures have been discovered [3–7]. Of particular interest is the paper [3] by H. S. Shapiro which presents a method for generating an evidently large number of extremal signatures.

In this paper, we consider a way to construct a collection of extremal signatures for polynomials in two variables. Also, by constructing certain product measures, we show how to obtain some additional extremal signatures.

2. Basic Results

The space of polynomials of degree n in k real variables will be denoted by P_n^k. In particular, a polynomial $p \in P_n^2$ has the form

$$p(x, y) = \sum_{k,s \geqslant 0} c_{ks} x^k y^s,$$

where $k + s \leqslant n$ and the c_{ks} are real numbers. By $R(\alpha)$ we shall denote the rotation of the plane

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Now, for given integers $n \geqslant 1$ and $m \geqslant 1$, define $U_{n,m}$ to be the set of all $p \in P_n^2$ such that $p \circ R(2\pi/m) = p$. Thus, $U_{n,m}$ consists of all polynomials of degree n which are invariant under a rotation through $2\pi/m$. We observe that $U_{n,m}$ is a linear subspace of P_n^2.

Let S_m be a sector of the plane with vertex angle $2\pi/m$, which in polar coordinates we write as $S_m = \{(r, \varphi) : 0 \leqslant \varphi < 2\pi/m\}$. Our approach to constructing extremal signature is based on

LEMMA 2.1. *Let σ be a signature on S_m which is extremal with respect to $U_{n,m}$. Then the signature*

$$\sigma_* = \operatorname{sgn}\left(\sum_{k=0}^{m-1} \sigma \circ R(2\pi k/m)\right)$$

is extremal with respect to $P_n{}^2$. The support of σ_* consists of all points with polar coordinates of the form $(\rho, \theta + 2\pi k/m)$ where (ρ, θ) is a point in the support of σ, and k is an integer.

Proof. Suppose $p \in P_n{}^2$ is such that $p\sigma_* > 0$ at all points in the support of σ_*. Let

$$p_* = \sum_{k=0}^{m-1} p \circ R(2\pi k/m).$$

Then $p_* \in U_{n,m}$. From the definition of σ_* we see that all points in the support of σ_* have the form $(\rho, \theta + 2\pi k/m)$, where (ρ, θ) is the support of σ. It follows that $p_*\sigma_* > 0$ on the support of σ_*. In particular, for the points in the support of σ, we have $p_*\sigma > 0$, which is a contradiction.

Thus, we can construct extremal signatures for $P_n{}^2$ by constructing signatures on S_m which are extremal with respect to $U_{n,m}$. To implement this idea, we must know the form of the polynomials in $U_{n,m}$. For any $p \in P_n{}^2$, we can write

$$p(r\cos\varphi, r\sin\varphi) = \sum_{k=0}^{n} r^k t_k(\varphi), \tag{2.1}$$

where t_k is a trigonometric polynomial of degree $\leqslant k$. If $p \circ R(2\pi/m) = p$, then each t_k is periodic with period $2\pi/m$. Thus, t_k has the form

$$t_k(\varphi) = \sum_{\nu=0}^{[k/m]} a_\nu^{(k)} \cos \nu m\,\varphi + b_\nu^{(k)} \sin \nu m\,\varphi,$$

and therefore we can rewrite expression (2.1) as

$$p = \sum_{k=0}^{[n/m]} p_k^{(1)}(r^2)\, r^{km} \cos km\,\varphi + p_k^{(2)}(r^2)\, r^{km} \sin km\,\varphi, \tag{2.2}$$

where $p_k^{(1)}$ and $p_k^{(2)}$ are polynomials of degree at most $[(n - km)/2]$. Hence, a polynomial p belongs to $U_{n,m}$ if and only if p has the form (2.2). We observe that if n/m is not too large, the expression (2.2) is relatively simple, and thus it is not difficult to determine the signatures on S_m which are extremal with respect to $U_{n,m}$.

3. The Case $n \leqslant m - 1$

Suppose we select the integers n and m such that $n \leqslant m - 1$. From (2.2), the polynomials in $U_{n,m}$ have the form $p(r^2)$, where $r^2 = x^2 + y^2$ and $p \in P^1_{[n/2]}$. Now, it is easy to show that any signature σ on S_m which is extremal with respect to $U_{n,m}$ must be one of the following two types:

I. The support of σ consists of two points (r, φ_1) and (r, φ_2) in S_m where $r > 0$ and $\sigma(r, \varphi_1)\, \sigma(r, \varphi_2) = -1$.

II. The support of σ consists of $N = [n/2] + 2$ points $(r_i, \varphi_i) \in S_m$ where

$$0 \leqslant r_1 < r_2 < \cdots < r_N \qquad \text{and} \qquad \sigma(r_i, \varphi_i)\, \sigma(r_{i+1}, \varphi_{i+1}) = -1.$$

The extremal signatures for P_n^2 corresponding to the signatures of type I consist of at least $2n + 2$ points with alternating signs on the circumference of a circle. Hence, these extremal signatures are a special case of a well-known extremal signature [3] for P_n^2. However, some of the extremal signatures for P_n^2 corresponding to the signatures of type II are new. For example, consider the particular extremal signature in Fig. 1 which is obtained in the case $n = 2$, $m = 3$. This signature bears a slight resemblance to a signature constructed by Collatz [5]; however, neither signature is a special case of the other.

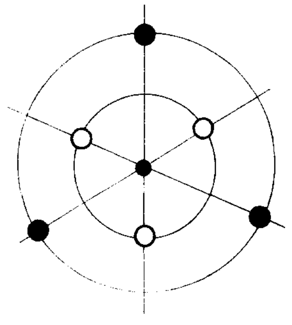

Fig. 1. Extremal signature of type II for $n = 2$ and $m = 3$.

One can show by a direct calculation that the extremal signature here can not be realized by the method of [3].

A large variety of extremal signatures for P_n^2 can be constructed from the type II signatures on S_m. It appears that most of these signatures can not be realized by the method of [3]; however, we have not yet completely investigated this question.

4. Some More General Extremal Signatures

In general, the extremal signatures obtained here are invariant under a rotation through the angle $2\pi/m$. It would seem, however, that this restriction could be relaxed in some cases. In this regard, we consider now a type of extremal signature which in a special case can be derived by our method, but which in general is not invariant under a rotation through $2\pi/m$.

We will use the following notation. If σ denotes a signature, then σ^+ will denote the set of points at which $\sigma = +1$ and σ^- will denote the set of points at which $\sigma = -1$.

Let $n \geqslant 1$ be an integer, and let φ_i for $i = 1, 2,..., n + 1$ be any set of angles with $0 = \varphi_1 < \varphi_2 < \cdots < \varphi_{n+1} < \pi$. Set $\varphi_{i+n} = \varphi_i + \pi$ for each i. Next, let r_i for $i = 1, 2,..., N = [n/2] + 1$ be radii such that $0 < r_1 < r_2 < \cdots < r_N$. Now, using the polar coordinate system (r, φ), define the signature σ_n by

$$\sigma_n^+ = \{r = 0\} \cup \{(r_i, \varphi_j) : i \text{ and } j \text{ are even}\},$$
$$\sigma_n^- = \{(r_i, \varphi_j) : i \text{ and } j \text{ are odd}\}.$$

This signature is illustrated in Fig. 2 for the cases $n = 4$ and $n = 5$.

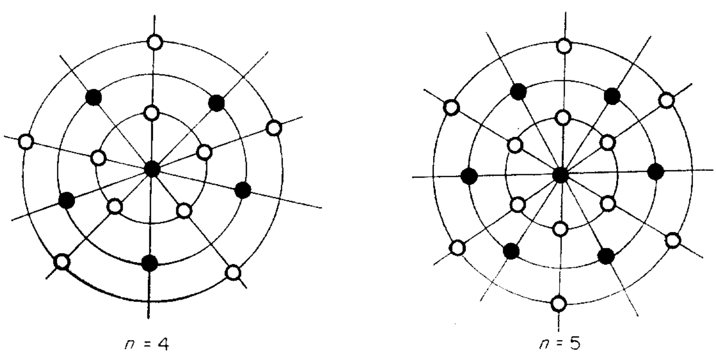

Fig. 2. The signature σ_n for $n = 4$ and $n = 5$.

For the special case in which $\varphi_j = \pi j/n + 1$, $j = 1, 2,..., 2n + 2$, σ_n can be constructed from the type II signatures mentioned in the previous section. However, in the general case, we have

THEOREM 4.1. *For $n \geqslant 1$, the signature σ_n is extremal with respect to P_n^2.*

Proof. It suffices to construct a measure α such that (a) the carrier of α is contained in the support of σ_n, (b) at each point in the carrier of α, the sign of α equals the sign of σ_n, and (c) for all $p \in P_n^2$, $\int p \, d\alpha = 0$.

Let us assume first that n is even. We construct the measure α in three steps.

Step 1. Define the signatures e_1 and e_2 on the real line by

$$e_1(r_i) = (-1)^i \qquad \text{for} \quad i = 1, 2, \ldots, [n/2] + 1; \quad e_1 = 0 \text{ otherwise,}$$

and

$$e_2(-r_i) = (-1)^i \qquad \text{for} \quad i = 1, 2, \ldots, [n/2] + 1; \quad e_2 = 0 \text{ otherwise.}$$

Then the signature which has value $+1$ at $r = 0$ and is defined elsewhere by $e_1 + e_2$ is extremal with respect to P_n^1. Hence, there are positive measures μ^+ on e_1^+, μ^- on e_1^-, λ^+ on e_2^+, and λ^- on e_2^- such that

$$p(0) + \int p \, d\mu^+ + \int p \, d\lambda^+ = \int p \, d\mu^- + \int p \, d\lambda^-$$

for all $p \in P_n^1$.

Step 2. The signature Λ defined by

$$\Lambda(\varphi_i) = (-1)^i \qquad \text{for} \quad i = 1, 2, \ldots, 2n + 2, \quad \Lambda = 0 \text{ otherwise,}$$

is extremal with respect to the space of trigonometric polynomials of degree n. Hence there are positive measures ν^+ on Λ^+ and ν^- on Λ^- such that

$$\int t(\varphi) \, d\nu^+(\varphi) = \int t(\varphi) \, d\nu^-(\varphi)$$

for all trigonometric polynomials $t(\varphi)$ of degree n. We will make the normalization

$$\int d\nu^+ = \int d\nu^- = 1.$$

Step 3. Now, let α be the measure on the plane with finite carrier which assigns measure $+1$ to the origin $r = 0$, and which is defined elsewhere by

$$d\mu^+(r) \, d\nu^+(\varphi) + d\lambda^+(r) \, d\nu^-(\varphi) - d\mu^-(r) \, d\nu^-(\varphi) - d\lambda^-(r) \, d\nu^+(\varphi). \quad (4.1)$$

It is easy to check that α has properties (a) and (b) mentioned above. As for property (c), let us consider each term in (2.1) separately:

1. When $k = 0$, the term has the form $p(r)$ where $p \in P_n^1$. Then

$$\int p\, d\alpha = p(0) + \int p\, d\mu^+ + \int p\, d\lambda^+ - \int p\, d\mu^- - \int p\, d\lambda^- = 0.$$

2. When $k \geqslant 1$, the term has the form $p(r)\, t(\varphi)$ where $p \in P_n^1$ and $p(0) = 0$, and t is a trigonometric polynomial of degree $\leqslant n$. Hence

$$\begin{aligned}\int p(r)\, t(\varphi)\, d\alpha &= \left(\int p\, d\mu^+\right)\left(\int t\, d\nu^+\right) + \left(\int p\, d\lambda^+\right)\left(\int t\, d\nu^-\right)\\ &\quad - \left(\int p\, d\mu^-\right)\left(\int t\, d\nu^-\right) - \left(\int p\, d\lambda^-\right)\left(\int t\, d\nu^+\right)\\ &= \left(p(0) + \int p\, d\mu^+ + \int p\, d\lambda^+ - \int p\, d\mu^- - \int p\, d\lambda^-\right)\left(\int t\, d\nu^-\right)\\ &= 0.\end{aligned}$$

Thus, the measure α has the desired properties which completes the proof when n is even. If n is odd, the same proof will hold, provided we replace the term $d\lambda^+(r)\, d\nu^-(\varphi)$ in (4.1) by $d\lambda^+(r)\, d\nu^+(\varphi)$, and we replace $d\lambda^-(r)\, d\nu^+(\varphi)$ by $d\lambda^-(r)\, d\nu^-(\varphi)$.

From this proof, we can obtain other extremal signatures by using an idea of Newman and Shapiro [6]. In steps 1 and 2 we had measures μ, λ, and ν such that

$$p(0) + \int p\, d\mu + \int p\, d\lambda = 0$$

for all $p \in P_n^1$, and

$$\int t\, d\nu = 0,$$

for all trigonometric polynomials $t(\varphi)$ of degree $\leqslant n$. Now, let $\mu = \mu_1 - \mu_2$, $\lambda = \lambda_1 - \lambda_2$ and $\nu = \nu_1 - \nu_2$ be arbitrary decompositions of μ, λ, ν, and let α be the measure which assigns measure $\int d\nu_1$, to the origin $r = 0$ in the plane, and which is defined elsewhere by

$$d\mu_1\, d\nu_1 + d\lambda_1\, d\nu_2 - d\mu_2\, d\nu_2 - d\lambda_2\, d\nu_1 .$$

Then it follows that

$$\int p\, d\alpha = 0$$

for all $p \in P_n^2$. Thus, the measure α determines an extremal signature for P_n^2. Some extremal signatures which were obtained in this fashion are shown in Fig. 3.

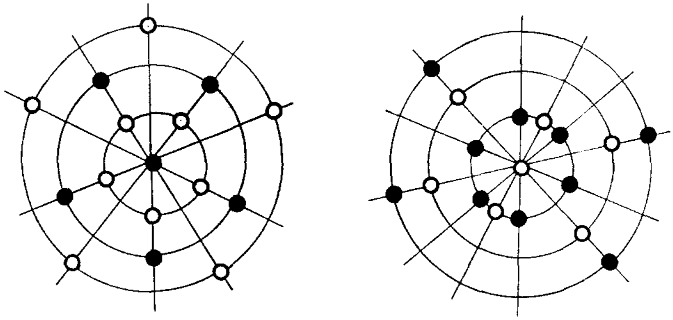

FIG. 3. Extremal signatures obtained by product measures.

In general, product measures can be used to obtain many extremal signatures for polynomials [6, 3]. In [6], this approach was used for polynomials written in a cartesian product form. Clearly, the underlying ideas in [6] can be applied to polynomials written in the form (2.1). For example, suppose $\mu(r)$ and $\nu(\varphi)$ are measures with finite carriers on $r \geqslant 0$ and $0 \leqslant \varphi < 2\pi$, respectively, such that

$$p(0) + \int p(r)\, d\mu(r) = 0,$$

for all $p \in P_n{}^1$ and

$$\int t(\varphi)\, d\nu(\varphi) = 0,$$

for all trigonometric polynomials of degree n. Let σ be the measure which is $+1$ at the origin and is given elsewhere by $d\mu_1\, d\nu_1 - d\mu_2\, d\nu_2$, where $\mu = \mu_1 - \mu_2$, $\nu = \nu_1 - \nu_2$. Then $\int p\, d\sigma = 0$ for all $p \in P_n{}^2$, so that σ determines an extremal signature for $P_n{}^2$ [Fig. 4(a)].

As another example, suppose the measures $\mu(r)$ and $\nu(\varphi)$ are selected so that

$$\int t(\varphi)\, d\nu(\varphi) = 0,$$

for all trigonometric polynomials $t(\varphi)$ degree $\leqslant k$, where $0 \leqslant k \leqslant n$, and

$$\int r^s\, d\mu(r) = 0,$$

for $s = k + 1, k + 2,..., n$. Then the measure $d\sigma = d\mu(r)\, d\nu(\varphi)$ determines an extremal signature for $P_n{}^2$ [Fig. (b)].

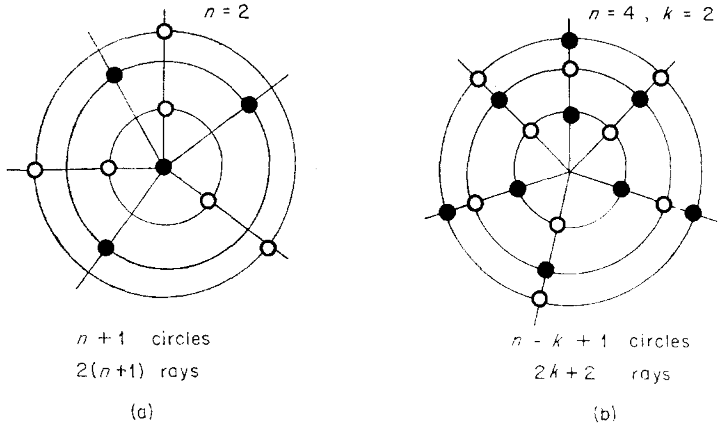

FIG. 4. Other extremal signatures obtained by product measures.

5. THE CASE $n = m + 1$

As one further illustration of our approach to constructing extremal signatures, let us consider the case $n = m + 1$. Assuming $m \geqslant 2$, the polynomials in $U_{n,m}$ have the form

$$c_1 r^m \cos m\varphi + c_2 r^m \sin m\varphi + p(r^2) \tag{5.1}$$

where c_1 and c_2 are real numbers and $p \in P^1_{[n/2]}$. Thus, in the sector $\{(r, \varphi) : 0 \leqslant \varphi < 2\pi/m\}$, we are searching for extremal signatures for the polynomials (5.1).

Let T denote the space of trigonometric polynomials of form $c_1 \cos \varphi + c_2 \sin \varphi$, and let V_n be the space of functions $p(r^2)$ where $p \in P^1_{[n/2]}$. Since the variable r is restricted to be nonnegative, it is easy to see that V_n satisfies the Haar condition. The polynomials (5.1) can now be written in the form $r^m t(m\varphi) + v(r)$ where $t \in T$ and $v \in V_n$. It suffices, therefore, to find the extremal signatures on the set $X = \{(r, \varphi) : r \geqslant 0,\ 0 \leqslant \varphi < 2\pi\}$ with respect to functions of the type

$$r^m t(\varphi) + v(r), \tag{5.2}$$

where $t \in T$ and $v \in V_n$.

Suppose now that $\sigma(r, \varphi)$ is a measure with finite carrier in X such that

$$\int_X [r^m t(\varphi) + v(r)]\, d\sigma = 0 \tag{5.3}$$

for all $v \in V_n$ and $t \in T$. For definiteness, assume the carrier of σ is contained in the lines $\varphi = \varphi_1, \ldots, \varphi = \varphi_N$ in X, and set $\mu_i(r) = \sigma(r, \varphi_i)$ for $i = 1, 2, \ldots, N$. Then (5.3) becomes

$$0 = \sum_{i=1}^{N} t(\varphi_i) \int r^m \, d\mu_i(r) + \int v(r) \, d\alpha(r),$$

where $\alpha = \mu_1 + \mu_2 + \cdots + \mu_N$. Taking $t = 0$ in this equation, we find that

$$\int v(r) \, d\alpha(r) = 0 \tag{5.4}$$

for all $v \in V_n$. It follows, therefore, that

$$\sum_{i=1}^{N} t(\varphi_i) \int r^m \, d\mu_i(r) = 0 \tag{5.5}$$

for all $t \in T$. Thus, we have two properties, (5.4) and (5.5) which σ must satisfy in order for (5.3) to hold.

On the other hand, suppose $\varphi_1, \ldots, \varphi_N$ are arbitrary distinct points in $[0, 2\pi)$ and $\lambda_1, \ldots, \lambda_N$ are real numbers such that

$$\sum_{i=1}^{N} t(\varphi_i) \, \lambda_i = 0 \tag{5.6}$$

for all $t \in T$. Further, let $\alpha(r)$ be a measure with finite carrier on $r \geqslant 0$ such that

$$\int v(r) \, d\alpha(r) = 0 \tag{5.7}$$

for all $v \in V_n$. We wish to find measures $\mu_1, \ldots, \mu_N$ such that

$$\int r^m \, d\mu_i(r) = \lambda_i \qquad i = 1, 2, \ldots, N,$$

and

$$\mu_1 + \mu_2 + \cdots + \mu_N = \alpha.$$

Clearly, in order for such measures μ_i to exist, it is necessary that

$$\int r^m \, d\alpha = \sum_{i=1}^{N} \lambda_i . \tag{5.8}$$

However, this condition is also sufficient. Indeed, construct measures $\mu_1, \ldots, \mu_{N-1}$ with finite carrier on $r \geqslant 0$ so that

$$\int r^m \, d\mu_i(r) = \lambda_i$$

for $i = 1, 2, \ldots, N-1$, and set

$$\mu_N = \alpha - \sum_{i=1}^{N-1} \mu_i \, .$$

Then μ_N is a measure with finite carrier on $r \geqslant 0$, and it follows immediately from (5.8) that

$$\int r^m \, d\mu_N(r) = \lambda_N \, .$$

Hence, $\mu_1, \ldots, \mu_N$ satisfy Eqs. (5.4) and (5.5), and therefore the measure $\sigma(r, \varphi)$ on X given by $\sigma(r, \varphi_i) = \mu_i(r)$ for $i = 1, 2, \ldots, N$ and $\sigma(r, \varphi) = 0$ otherwise, determines an extremal signature on X with respect to the functions (5.2).

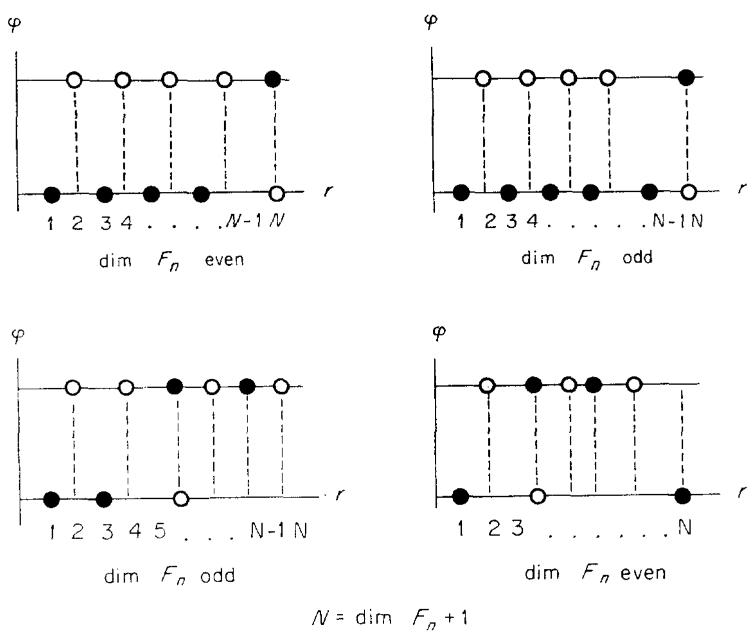

FIG. 5. Extremal signatures on X for the case $n = m + 1$.

From this development, it follows that we can obtain all the extremal signatures on X with respect to the functions (5.2) by constructing the extremal signatures corresponding to the measures $\mu_1, ..., \mu_N$. A particularly simple way to choose $\lambda_1, ..., \lambda_N$ is to require that (5.6) hold for all trigonometric polynomials of degree one. Then the "compatibility condition" (5.8) becomes

$$\int r^m \, d\alpha(r) = 0. \tag{5.9}$$

Now, let F_n be the space of functions on $r \geqslant 0$ with the form $ar^m + v(r)$, where $v \in V_n$. Using Descartes' rule of signs, it is not difficult to see that F_n satisfies the Haar condition for each n. Hence, since α is characterized by properties (5.7) and (5.9), any extremal signature corresponding to α contains $K_n + 1$ points with alternating signs, where K_n is the dimension of F_n. We observe that $K_n = [n/2] + 1$ if n is odd, and $K_n = [n/2] + 2$ if n is even.

It is clear that many extremal signatures can be obtained in the case $n = m + 1$. In Fig. 5, we illustrate only a few of them. For convenience, we have taken $N = 2$ and $\lambda_1 = \lambda_2 = 0$. Even in this special case, there are many extremal signatures.

6. Conclusion

An analysis similar to the one in the previous section can be performed for the case $n = m + k$ where $k < m$. In general, however, our "rotation" technique for constructing extremal signatures becomes somewhat difficult to implement when the ratio n/m is very large. Even so, a large variety of extremal signature for P_n^2 can be obtained by this method.

All of our extremal signatures are invariant under some rotation of the plane. However, as hinted in Section 4, this restriction is probably not essential.

We suspect that many of our extremal signatures cannot be realized by the method of [3], but this question requires further investigation.

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